A STRICTLY SPECIAL-RELATIVISTIC DISCUSSION OF EHRENFEST PARADOX AND SAGNAC EXPERIMENT SUGGESTS ANOTHER POSSIBLE EXPERIMENTAL FALSIFICATION OF SPECIAL RELATIVITY

Abstract - We suggest to utilize Sagnac effect in order to introduce a possible definition of the "proper length" L' of a rotating circumference C', and the phenomenon of relativistic aberration in order to define the "proper diameter" D' of C', in such a way that L'/D' = π, and Ehrenfest paradox disappears. At last, an "experiment" is proposed: is it possible to test light's relativistic aberration in a terrestrial laboratory?

Résumé - On propose d'utiliser l'effect de Sagnac pour mesurer la "longueur propre" L' d'une circonférence roulante, et l'aberration relativistique pour mesurer le "diamètre propre" D' de C'. L'on obtient enfin que L'/D' = π, et le paradoxe d'Ehrenfest va donc disparaître. Enfin, on propose un'*expérience*: est-il possible de vérifier en laboratoire l'aberration relativistique de la lumière?

Key words - Special Relativity, Rotating Platform, Accelerated Observers, Sagnac Effect, Light's Aberration.

1 - A correct statement of the paradox

It was already in 1909 that Paul Ehrenfest[(1)] pointed out the difficulties that Special Relativity (SR) has in the definition of "rigid body", by means of his famous question concerning the length's variation of a rotating circumference. Since then, countless attempts of "explanation" have been given, and to the whole argument of the "rotating platform" some attention is dedicated until today[(2),(3)], notwithstanding it has been apparently set-up by deeper analyses of SR, and rigorous foundations of the concept of "relativistic rigid body"[(4),(5)]. Many of these explanations introduce General Relativity, or deal with the *concrete physical structure* of the spinning disk, then introducing terms as: "elastic dilations", "radial stresses", "bendings of the disk's surface", etc.. These arguments appear to the present author rather unsatisfactory, since they fail to give a "solution" of the question in the same conceptual frame in which it was expressed, namely in a *purely geometrical-kinematical* set-up[1].

First of all, let us state Ehrenfest's *dilemma*, in the same "rough terms" which are currently used in its popularizations[(7),(8)].

In an inertial reference frame (IRF) Ω in the 3-dimensional Minkowski space-time **M**, whose coordinates are $(x,y,t)^2$, let us introduce a circular platform P' of radius R, rotating in Ω at some angular speed ω different from zero, but whose centre $A \equiv (0,0,t)$ is at rest in Ω. Then, think of an (accelerated) "observer" A' placed in the rim C' of P', whose speed with respect to (from now on: wrt) Ω will be $v = \omega R$. We can suppose for instance that A' is defined by the motion equations:

$x = R\cos(\omega t)$, $y = R\sin(\omega t)$

(all physical and mathematical quantities introduced above are defined wrt the given IRF, which is indeed a "privileged" IRF wrt P').

Well, if L' is the length of C' *as seen by A'* (in this paradox, L' will play the role of the "true" length of C', or of the "proper" length of C'), what is the relation between the length L of C' in Ω, and L'?

In order to answer to this question, one generally suggests to "regularly" divide C' in n parts C'_i , i=0,...,n-1 , each one of "proper length" L'_i , in such a way that $L' = \Sigma(L'_i)$. Then, *in a given instant t** (once again, wrt Ω!), if n is large enough, one could think that each one of these C'_i could be approximatively considered as "inertial", at least in a "small" neighborhood of t*. Let us call $\Omega_i(t^*)$ the IRFs in which these C'_i can be considered *at rest*, in that instant t* . Then, because of the well known *relativistic length contraction*, in the passage from $\Omega_i(t^*)$ to Ω, each length L_i of C'_i wrt Ω should be equal to the corresponding proper length L'_i of C'_i in $\Omega_i(t^*)$, multiplied by the *shrinking factor* $\sqrt{(1-v^2)}$. Thus one would get: $L_i = L'_i\sqrt{(1-v^2)}$, and this would imply, since $L = \Sigma(L_i)$:

$$L = \Sigma(L_i) = \Sigma[L'_i\sqrt{(1-v^2)}] = L'\sqrt{(1-v^2)} \qquad (1) .$$

Furthermore, if we introduce the *proper radius* R' of C', or of P' (once again, the radius of the platform *as seen by A'*), we should have no length contraction at all for R', since the radius's motion is *transversal* to the motion of all $\Omega_i(t^*)$ (as usual, wrt Ω), and then:

$$R' = R \qquad (2) .$$

From (1) and (2), Ehrenfest paradox follows immediately, since one should expect:

$$L = 2\pi R \, , \; L' = 2\pi R' \;\Rightarrow\; L = L' \quad \text{(applying (2))} \qquad (3)$$

which is incompatible with (1).

The first solution of the paradox consists in rejecting the second identity in (3). One suggested to write instead: $L' = 2\pi' R'$, where π' is the *convenient* value of "π" in the "geometry of the rotating platform". It would then follow, from (1), that:

$$L = 2\pi R = L'\sqrt{(1-v^2)} = 2\pi' R'\sqrt{(1-v^2)} \Rightarrow \pi = \pi'\sqrt{(1-v^2)} .$$

In conclusion, the value of π in the rotating platform's geometry would seem to be **different** - and *greater* - from the "ordinary" one. This was interpreted by asserting that the geometry of P' should be considered, in some sense, a *non-euclidean geometry*[3].

Let us start our purely "geometrical" discussion of this riddle showing how the common argument which implies (1) could be rejected. As a matter of fact, the previous deduction of this identity can be criticized at least at a double level of understanding.

The first one, is that those "proper lengths" L'_i of C'_i , namely the lengths of C'_i as seen by A' - whatever this expression could exactly mean! - do not necessarily coincide with the lengths Λ_i of the C'_i in the IFRs $\Omega_i(t^*)$. These C'_i are indeed not at rest wrt **any** IFR containing another one of them, but are moving at different speeds. For this reason, leaving for the moment apart problems of synchronization[4], if C'_0 is that part of C' corresponding to A' itself, and $\Omega_0(t^*)$ the associated IFR, all proper lengths Λ_i , $i = 1,...,n-1$, will be seen by $\Omega_0(t^*)$ *contracted* by a shrinking factor $\sqrt{(1-v_i^2)}$ of the same kind as before, where v_i is now the relative speed between $\Omega_i(t^*)$ and $\Omega_0(t^*)$ (we shall come back to this argument in the next section). That is to say, the "rough" argument which led to (1) would have been correct only in the case of the famous *Einstein's train*, in which **all** passengers are at rest, one wrt to the other, *in the same* IFR. In the case of a rotating platform, on the contrary, this is no longer true, at least from the point of view of $\Omega_0(t^*)$, and here it comes the second level of understanding of the paradox.

The real problem one has to face, does concern the exact meaning of expressions like: "as seen by some observer", "measured in the platform system", and so on.

As a matter of fact, in order to define the "proper length" L', which is the *key* of the whole paradox, first of all one should discuss how length measures can be introduced in SR in a *general coordinate system*, "adapted" to some *observer field* X[5].

Because of the particular nature of "space" and "time" in relativity, *general* coordinates (X,Y,T) give spatial length measures of "objects", and "trajectories", when the ds^2 admits a *splitting* of the kind: $ds^2 = d\sigma^2 + gdT^2$, where $d\sigma^2$ is a *positive definite* quadratic form in (X,Y) (we can say that in this case the coordinate system is *orthogonal*)[6]. Moreover, one knows that it is impossible to find a *coordinate time* T which coincides with the *proper time* of *all* observers of X, unless X is an inertial field of observers (*geodesic* observers), and (X,Y,T) are the familiar Lorentz coordinates. That is to say, if one leaves the usual Lorentz coordinates in **M**, and introduces general ones (even orthogonal in the sense above specified), then the "time" in these systems cannot be always measured by "clocks"[7].

This proves that, if one wants to define the *proper length* we are investigating, first of all he must give up this requirement. But this is not the only trouble that one meets in trying to introduce a "suitable" coordinate system associated to the platform[8] (that is to say, a system in which all, or "most", observers in the platform P', or just in C', are at rest). In order to carry on a rigorous analysis, one has to introduce first the *observer field* U, defined as the field of the 3-velocities of all observers:

$$x = \rho\cos(\omega t+\theta),\ y = \rho\sin(\omega t+\theta)\ ,\ -\pi < \theta < \pi\ , \qquad (4)$$

for any ρ : $R-\varepsilon < \rho < R+\varepsilon$, and for some "small" ε[9]. Then, one must check whether is it possible, or not (perhaps even introducing some "restriction" of U, either depending on ρ, on θ, or on t), to find an orthogonal coordinate system adapted to this field U[10]. Well, each one of these 3-velocities, let us call it u, has indeed an *infinitesimal rest-space*, defined as the 2-dimensional linear subspace of **M** Lorentz-orthogonal to u, and one could think, to begin with, to define L' as the length of the section of this rest-space, in some given instant, with the cylinder: $x^2 + y^2 = R^2$, which is the surface corresponding to all observers on C'[11].

Making computations in our IFR Ω (that is to say, using the coordinates (x,y,t)), in the instant t = t*, the rest-space corresponding to the observer A' is defined by the equation:

$$-v\sin(\omega t^*)(x-R\cos(\omega t^*)) + v\cos(\omega t^*)(y-R\sin(\omega t^*)) - (t-t^*) = 0 \qquad (5)$$

and the section we are looking for is the ellipse of parametric equations:
$\lambda = R\cos(u)$, $\mu = R\sin(u)$,
in the 2-dimensional linear sub-space of parametric equations:
$x = \lambda$, $y = \mu$, $t = -v\sin(\omega t^*)\lambda + v\cos(\omega t^*)\mu + t^*$.

These equations give for the quadratic form $d\sigma^2$ the expression:
$$d\sigma^2 = g_{11}d\lambda^2 + 2g_{12}d\lambda d\mu + g_{22}d\mu^2 = (1-v^2\sin^2(\omega t^*))d\lambda^2 +$$
$$+ 2v^2\sin(\omega t^*)\cos(\omega t^*)d\lambda d\mu + (1-v^2\cos^2(\omega t^*))d\mu^2\ .$$

The required length L' is then given by the formula:

$$L' = R\int_{[0,2\pi]}\sqrt{[(1-v^2\sin^2(\omega t^*))\sin^2(u)+}$$

$$-2v^2\sin(\omega t^*)\cos(\omega t^*)\sin(u)\cos(u)+(1-v^2\cos^2(\omega t^*))\cos^2(u)]du =$$
$$= R\int_{[0,2\pi]}\sqrt{(1-v^2\cos^2(u-\omega t^*))}du = R\int_{[0,2\pi]}\sqrt{(1-v^2\cos^2(u))}du .$$

This identity shows at least that L' should indeed rather be *smaller* than *greater* than $L = 2\pi R$, which is in better qualitative agreement with our previous considerations. For instance, for $v << 1$ (remember that, in the actual notations, $v = v/c$), one gets the following approximation, up to second order in v:

$$L' \approx R\int_{[0,2\pi]}[(1-v^2\cos^2(u)/2]du = R(2\pi-\pi v^2/2) = 2\pi R(1-v^2/4) ,$$

which is quite different from the analogous one coming from (1):

$$L' \approx 2\pi R/(1-v^2/2) \approx 2\pi R(1+v^2/2) \qquad (6) .$$

Apart that, the previous value of L' cannot be truly endowed with any "physical meaning", because, due to the *curvature* of A', the rest-spaces (5) (when t* varies), are not *parallel* in **M**, and of course this is "bad", since we want that these spaces would correspond to "simultaneity spaces": T = constant, for some coordinate time T. This implies that one should at least exclude the intersection of these spaces from any possible domain of the coordinate system we are looking for (at least in the case of *flat* rest-spaces), and this would not have painful consequences for our purposes only if these intersections, which are lines, would not meet our cylinder, but a straightforward computation shows that this is not the case!

The question becomes then to see whether these rest-spaces could be *integrated* in order to give a *global* rest-subvariety, possibly not a flat one, but a straightforward computation shows that the vector field U (or any one of its "restrictions") is not *irrotational*, and this is unfortunately a necessary and sufficient condition for the required integrability[12]. In other words, it is impossible to find a coordinate system, with the required properties, which would "include" the whole, or even part, of C'.

At this point one could think of playing other "tricks", for instance to give up the demand of including in the coordinate system we are looking for "too many" rotating observers on P', and just to concentrate on C', trying to find a suitable "extension" of the vector field above (first restricted to the circumference C', that is to say, for $\rho = R$). Even in this case, one finds difficulties[13], as one would meet even trying to introduce any "proper" measure of the *radius* of C' (of P'), but let us pass over this point[14]. The truth is that it is not possible to univocally introduce in SR any "coordinate system associated to the platform"[15], and that *direct*[16] rigorous definitions of both L' and R' are impossible.

Summing up, one could possibly reject (which is in some sense a "solution"!) Ehrenfest's argument, saying that L' and R' cannot be suitably (univocally) defined in SR. But instead of doing that, we think rather instructive, instead, to *look for a possible alternative definition* of these quantities. We shall show that it seems plausible to introduce these definitions in such a way that the relation between L and L' should be the "inverse" of (1) (and (6)), namely:

$$L' = L\sqrt{(1-v^2)} \approx L(1-v^2/2) = 2\pi R(1-v^2/2) \qquad (7)$$

and that the relation between R and R' could be, instead of (2):

$$R' = R\sqrt{(1-v^2)} \qquad (8)\ .$$

The first shrinking will be interpreted as a consequence of *time dilation*, rather than of length contraction, and the second one of *relativistic light's aberration*. In conclusion, there would not be an Ehrenfest paradox anymore, since, with the suggested definitions:

$L'/R' = L/R = 2\pi$!

2 - A possible connection with Sagnac experiment

The definition we are looking for, will be obtained by introducing another widely discussed argument, the *Sagnac experiment*[(10)], which even nowadays somebody believes, but erroneously, a confutation of SR[(11)].

As a matter of fact, even if it is never part of a suitable coordinate system in which it is at rest, A could obtain a measure of L', *just by means of its own unique clock*, in the following way. Suppose that A' sends, at some instant τ_0 of its proper time, two light's beams along C', in the two opposite directions. The two beams will cover all the length of C', and then will come back to A', of course *not simultaneously*, both wrt A' and wrt Ω. From Ω's point of view, the computation is quite easy: the forward beam will come back to A' after a time interval: $\Delta t_F = 2\pi R/(1-v)$, the backward beam after a time interval: $\Delta t_B = 2\pi R/(1+v)$.

The *ratio* k between these two time intervals, $k = \Delta t_F/\Delta t_B$, is the so-called *Sagnac effect*, and it is actually a quantity greater than 1, depending on the speed v of A' wrt Ω (or, which is the same, wrt the centre A of the platform).

From the point of view of the proper time τ of A'[17], the two corresponding proper time intervals, $\Delta\tau_F$ and $\Delta\tau_B$, will be equal to:

$$\Delta\tau_F = \Delta t_F\sqrt{(1-v^2)}\ ,\quad \Delta\tau_B = \Delta t_B\sqrt{(1-v^2)} \qquad (9)\ .$$

These identities imply that: *the Sagnac effect k is the same*, either as seen by A, or as seen by A'.

As a consequence, A' can indeed realize that the platform is rotating, with no contradiction at all with the I Postulate of SR[18], and can even measure its "absolute" rotational speed (namely, wrt the centre A):

$$v = (k-1)/(k+1) \qquad (10) .$$

Well, from (9), and from the above values for Δt_F and Δt_B, we have:

$$\Delta\tau_F = [2\pi R\sqrt{(1-v^2)}]/(1-v) \ , \ \Delta\tau_B = [2\pi R\sqrt{(1-v^2)}]/(1+v) \qquad (11) .$$

From these identities, we could at last conclude that[19]:

* - $2\pi R\sqrt{(1-v^2)}$ *could* be defined as the "proper length" of C' (the length of C' as seen by A');

** - according to this definition, the *average* light's speeds, again wrt A', forward and backward, are respectively equal to the "classically expected values": c-v and c+v[20].

Remark 1 - It is perhaps curious to point out that precisely this same reasoning gives also a physical argument in favour of choice (1)! Introducing the so called "radar method", one could *define* the distance of something X which is far away from any given observer Y as the half of the Y-proper time that light spends in order to go from Y to X, and then to come back (remember that actually c = 1). If we use this definition in the case of the rotating circular platform, we find that the proper time interval that a light's beam spends going from A' to A' itself, along the circumference in some direction, plus the time which is spent going *back*, from A' to A' again, but in the other direction, is equal to:

$$[2\pi R\sqrt{(1-v^2)}]/(1-v) + [2\pi R\sqrt{(1-v^2)}]/(1+v) =$$
$$= [4\pi R\sqrt{(1-v^2)}]/(1-v^2) = 4\pi R/\sqrt{(1-v^2)}$$

and the half of this value is exactly (1).

It is rather important to remark that there are even **two** more independent arguments which can justify the previous *definition* *. The first one goes as follows. Think of a circumference C "strictly contiguous" to C' (same centre A, same radius R), but which *does not rotate* (in Ω) (C is quite a different observer field than C'!). Suppose to choose an "observer" B $\in$ C (with a slight abuse of notation!), which has the role of indicating to A' when it has made a whole rotation (when B comes back to A', as A' sees it!). From the point of view of the clock of B, the event: *coming back of A'*, will happen at time intervals equal to $2\pi R/v$. From the point of view of the clock of A', instead, by time dilation, the corresponding time intervals will be equal to $[2\pi R\sqrt{(1-v^2)}]/v$, which is perfectly compatible with our suggested definition (do not forget that A' is indeed able to evaluate its Ω-speed v by means of the Sagnac effect k)[21].

Remark 2 - The previous argument shows that the shrinking (7) could be interpreted even as nothing else but the "ordinary" length contraction of the proper length of C as seen by the (*really rotating*) oberver A'. That is to say, if A' has some trouble in measuring C', it has less difficulties in measuring C!

It is very instructive to show that the approximation, up to second order in v, given by * (see (7)): $L' \approx 2\pi R(1-v^2/2)$, can be deduced also with a quite different purely geometrical-kinematical procedure. The idea is to start from those single "local lengths" Λ_i introduced in section 1, and then to modify them according to our previous considerations, before proposing their sum as a possible value of the circumferences's length "as seen by an observer on it"[22].

To this purpose, let us introduce the following n observers on C:

$x = R\cos(\omega t+2\pi i/n),\ y = R\sin(\omega t+2\pi i/n)\ ,\ i = 0,1,...,n-1\ ,$

and for each one of these, in a given Ω-instant t = t*, the IFRs that we have called $\Omega_i(t^*)$. Then, the IFR $\Omega_0(t^*)$ will be the one associated to A', and in order to evaluate L' *as seen by A'*, as we have already said, we must add to the value $2\pi R/[n\surd(1-v^2)]$ (the length wrt $\Omega_0(t^*)$ of that part of circumference "near" A') the other n-1 analogous values $2\pi R/[n\surd(1-v^2)]$, each one multiplied by the shrinking factor $\surd(1-v_i^2)$, where v_i is the relative speed between $\Omega_i(t^*)$ and $\Omega_0(t^*)$:

$$L' = \lim \{\Sigma[2\pi R\surd(1-v_i^2)]/[n\surd(1-v^2)]\} \qquad (12)$$

where the sum is meant from i = 0 to i = n-1, and the limit is taken for $n \to \infty$.

In order to compute v_i , we must find the coordinate transformations F_i connecting $\Omega_0(t^*)$ and $\Omega_i(t^*)$, $F_i : (x°,y°,t°) \to (x^i,y^i,t^i)$. If one introduces the transformations:

$G_i : X^i = -x\sin(2\pi i/n)+y\cos(2\pi i/n),\ Y^i = x\cos(2\pi i/n)-y\sin(2\pi i/n)\ ,\ t = t$

$H_i : x^i = (X^i-vt)/\surd(1-v^2),\ y^i=Y^i,\ t^i=(t-vX^i)/\surd(1-v^2)\ ,$

then F_i is the product of the following ones (we can obviously confine ourselves to the case t* = 0):

$(H_0)^{-1} : (x°,y°,t°) \to (X^0,Y^0,t)\ ,\ (G_0)^{-1} : (X°,Y°,t) \to (x,y,t)\ ,$

$G_i : (x,y,t) \to (X^i,Y^i,t)\ ,\ H_i : (X^i,Y^i,t) \to (x^i,y^i,t^i)\ ;$

$F_i = H_i\, G_i\, (G_0)^{-1}\, (H_0)^{-1} :$

$x^i\surd(1-v^2) = (Y°-R)\sin(2\pi i/n) + X°\cos(2\pi i/n) - vt =$

$= (y°-R)\sin(2\pi i/n) + [(x°+vt°)\cos(2\pi i/n)]/\surd(1-v^2) - v(t°+vx°)/\surd(1-v^2)$

$y^i = (Y°-R)\cos(2\pi i/n) - X°\sin(2\pi i/n) + R =$

$= (y°-R)\cos(2\pi i/n) + [(x°+vt°)\sin(2\pi i/n)]/\surd(1-v^2) + R$

$t^i\surd(1-v^2) = t-v[(Y°-R)\sin(2\pi i/n) + X°\cos(2\pi i/n)] =$

$= (t°+vx°)/\surd(1-v^2) - v[(y°-R)\sin(2\pi i/n)+(x°+vt°)\cos(2\pi i/n)]/\surd(1-v^2)).$

From these equations one can get the origin's motion, wrt the time t°, and then the precise value of v_i . The corresponding rigorous expression is rather complicated, but one can at least compute its approximation, up to second order in v. One gets:

$$v_i \approx v(\cos(4\pi i/n)-\cos(2\pi i/n), \sin(4\pi i/n)-\sin(2\pi i/n)) \qquad (13)$$

and (12) becomes:

$$\begin{aligned} L' &= \lim \{\Sigma[2\pi R\sqrt{(1-v_i^2)}]/[n\sqrt{(1-v^2)}]\} \approx \\ &\approx \lim \{\Sigma[2\pi R\sqrt{(1-2v^2+2v^2\cos(2\pi i/n))}]/[n\sqrt{(1-v^2)}]\} \approx \\ &\approx \lim \{\Sigma[2\pi R(1-v^2+v^2\cos(2\pi i/n))]/[n(1-v^2/2)]\} \approx \\ &\approx \lim \{\Sigma[2\pi R(1-v^2+v^2\cos(2\pi i/n))(1+v^2/2)]/n\} \\ &\approx \lim \{\Sigma[2\pi R(1+v^2/2-v^2+v^2\cos(2\pi i/n))]/n\} = \\ &= 2\pi R(1-v^2/2) + 2\pi R v^2 \lim \{\Sigma[\cos(2\pi i/n)]/n\} , \end{aligned}$$

which implies exactly (7), as $\Sigma[\cos(2\pi i/n)]/n$ is an integral sum of the differential form cos(x)dx !

This could be the end of our discussion, since A' could ascribe the *apparent* shrinking (7) of C' (or of C) to its own time dilation, and then alter correspondingly its "longitudinal" length measures in such a way that no changes would be required at all (wrt to the corresponding measures in Ω). All the same, in the next section we shall discuss the question concerning the "transversal" length measures, once again *as seen by A'*. That is to say, we shall discuss also the validity of (2), which has been until now untouched!

3 - A possible connection with relativistic aberration

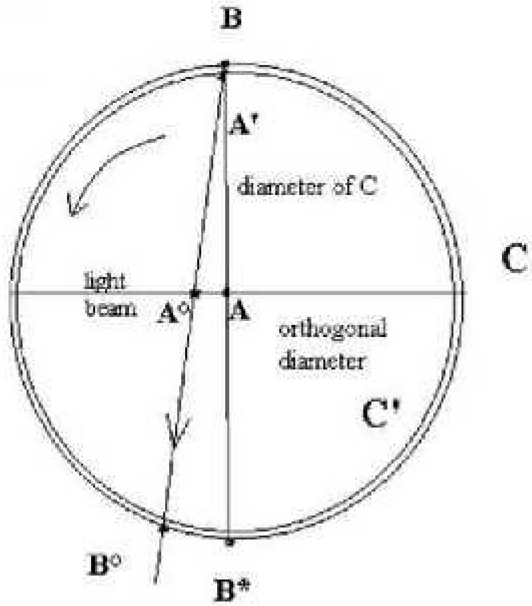

Figure 1: Light's aberration from the spinning source A', as seen by the fixed circumference C. θ is the angle between the two half-lines BB* and BB°.

We shall now examine another interesting feature of Ehrenfest paradox. One should really accept the identity R' = R ? Namely, that the *real* diameter D = 2R of C' (or of P', or of C), as measured in Ω (or by A), is equal to the *apparent* diameter of C', the one *seen by A'*?[23]

According to our previous comments, certain quantities cannot be "really" measured by A', and one should always perform a careful mathematical analysis in order to show whether or not some values could be precisely (and univocally) defined. Anyway, we can introduce in our "game" the following "physical" considerations. Since the Ω-diameter 2R of C is the distance between a "point" B in C and its *antipodal point* B*, 2R = d(B,B*), one could ask: which should be thought of as the "antipodal point" of B, *as seen by A'*? Would it be the same B* we have introduced before, or another point B°? Then, we shall compute the "transversal distance" between B and B° (just the ordinary distance in Ω, d(B,B°)), and we propose to call it the *apparent diameter* of C (we shall denote it by D').

Well, in order to answer to this question, we can suppose that, in some instant t*, for instance when A' is exactly in front of the previous B, A' sends a mono-directional light's beam (a photon) "towards the centre of the platform", namely in the orthogonal direction (in Ω[24]) to the tangent of C. Then, we could look for the point B° in C which would "receive" this photon. It is clear that, **according to SR**, the photon sent by A' does not pass through the centre of P, and does not "hit" C in the point B*. The reason for that is easily explained by means of *relativistic light's aberration*.

The starting point for understanding this phenomenon is to carefully distinguish between <u>speed</u> (*scalar* velocity) and <u>velocity</u> (*vectorial* velocity), which in some language is not possible[25]. With this specification, SR's II Postulate prescribes just that the *light's speed* is independent on the source's motion (in any inertial frame), but **not** the *light's velocity*, which in fact *can depend* on the source's velocity.

As a simple example, in a very common set-up, let us take a photon travelling backward along the y-axis (the photon is supposed to start at the time t = 0 from some indefinite distance L > 0):

ϕ : x = 0, y = L-t, z = 0 (velocity (0,-1,0), speed 1).

If you imagine the "usual" observer travelling along x-axis with some uniform velocity (v,0,0), endowed with a Lorentz coordinate system (x',y',z'), then you can use a Lorentz transformation in order to connect coordinates (x,y,z) and coordinates (x',y',z'):

$x = (x'+vt')/\sqrt{(1-v^2)}$, $y = y'$, $z = z'$, $t = (t'+vx')/\sqrt{(1-v^2)}$,

and then the motion of ϕ becomes, in these new coordinates:

$x = 0 \rightarrow (x'+vt') = 0 \rightarrow x' = -vt'$,

$y = L-t \rightarrow y' = L-(t'+vx')/\sqrt{(1-v^2)} \rightarrow y' = L-t'\sqrt{(1-v^2)}$,

$z = 0 \rightarrow z' = 0$.

These equations show that, in the new coordinates, the *photon's velocity* is $(-v,-\sqrt{(1-v^2)},0)$ (of course, the *photon's speed* is always 1, since: $v^2+(1-v^2) = 1$!), which clearly *does depend* on the velocity of the source (in the system (x',y',z'), this velocity is equal to (-v,0,0)).

This is the reason for relativistic aberration, since the light coming from the source will be received by the "moving observer" shifted under an angle θ such that $tg(\theta) = v/\sqrt{(1-v^2)} \approx v$ (up to second order terms in v)[26], and that is all.

Coming back to our case - and we repeat it, **according to SR** - the photon sent by A' will go into a straight line *only* with respect to the *virtual observer* which would go on, in a state of uniform motion, with the same velocity of A' in the very moment of the photon's emission. Thus, the photon will be "aberrated" in Ω, wrt the geometrical diameter δ of C starting from the point B, with an angle θ such that, as a simple geometrical argument shows (see figure 1):

$$\cos(\theta) = \sqrt{(1-v^2)} \qquad (14) .$$

As a matter of fact, if we call Δt the Ω-time employed by the photon in the part of its travel until it "cuts" the diameter orthogonal to δ, in the point A°, we obviously have $(\Delta t)^2 = (v\Delta t)^2 + R^2$, and then $\cos(\theta) = R/\Delta t = \sqrt{(1-v^2)}$, as asserted.

It is very easy to compute now which is the point B° on C that our photon would hit. Since the triangle BB°B* is a right triangle in B°, we will get at last:

$$D' = d(B,B°) = \text{apparent diameter of } C = \\ = 2R\cos(\theta) = 2R\sqrt{(1-v^2)} \qquad (15) .$$

From this identity, by means of (7), one would get, as announced: $L'/D' = \pi' = L/2R = \pi$, which could be interpreted as another possible "solution" of Ehrenfest paradox!

4 - A possible experimental falsification of Special Relativity

The above discussed relativistic difference between *velocity* and *speed* of light, and the corresponding correct understanding of the II SR Postulate, could perhaps be the conceptual ground for some attempt to compare SR predictions with analogous aether-theoretic expectations[27]. As a matter of fact, one could suppose that it would be natural, in an aether-frame, to have *total independence*, namely *vectorial independence*, of light's velocity on the velocity of the source (and not only independence of the *scalar* values).

One could think for instance to use the same circular platform P' of the previous discussion. First suppose P' at rest, and place a mono-directional (the most possible point-like!) photon source S in the rim C of the platform, near to a "fixed" point B, directed towards the centre A of P'. A "fixed" light's detector S* is placed near the antipodal point B* of B, and can detect the arrival of the photons emitted by S. Then, we can make the platform rotate, and arrange things in such a way that, in "stationary" conditions, S does emit a photon only when it is in front of B. At last, one could check whether S* will continue to detect these photons, even when the source is rotating, as an aether theory would foresee, or not. That is to say, whether light is really "dragged" by the velocity of the source, as SR would predict, or not.

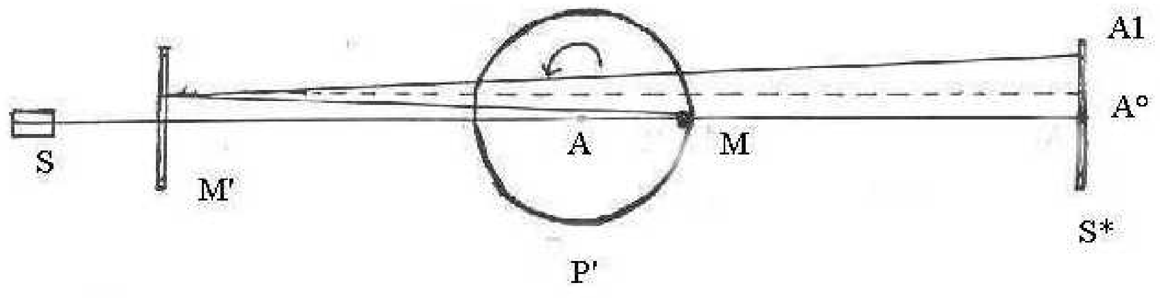

Figure 2: The proposed experiment, using a rotating circular platform P' of centre A, as seen from the top.
S = photon source
S* = mirror-screen detector
M' = semi-transparent mirror
M = reflecting mirror
A° = point in which the light's beam hits S* when there is not reflection by M
A^1 = point in which the light's beam hits S* after one reflection by M and M'

One could even think to take both detector S* and source S *fixed* in the laboratory, and to use instead as *moving source* an (almost possible point-like, as before!) mirror M placed in the rim C' of P'. S emits photons towards a point A' in S*, such that the line SA' contains the centre A of P', and these photons are reflected only when M passes through this line (that is to say, with a Ω-period equal to $2\pi R/v = 2\pi/\omega$)[28]. The backward photons can be detected by a fixed screen-detector S*. In order to avoid S and S* to be too much close one to the other, one could place a semi-transparent mirror M' at some distance L_1 from A (between S and A, see figure 2), orthogonal to the photon beam, in such a way that the emitted photons pass through M', and could then place S* at some distance L_2 from A, in order to detect the backward photons reflected by the "other" face of M'. One can even think to increase the researched effect, *by repeated reflections*, making use as detector S* of another "mirror" (in figure 2, A° = A' is the point in which the light's beam hits S* when there is no reflection by the mirror M, A^1 is the point in which the light's beam hits S* after one reflection by M and M', and so on).

It is very easy to give quantitative evaluations for this *Gedanken-Experiment*. For a given radius R, a given angular speed ω, the distance δx between the impact point A° of the forward non-aberrated photons on S*, and the impact point A^1 of possibly aberrated photons, after only one reflection, would be equal, in force of (14), to:

$$\delta x = (L_1+R)\mathrm{tg}(\theta) + (L_1+L_2)\mathrm{tg}(\theta) = (2L_1+L_2+R)\mathrm{tg}(\theta) =$$
$$= [(2L_1+L_2+R)\sqrt{(1-\cos^2(\theta))}]/\cos(\theta) = (2L_1+L_2+R)v/\sqrt{(1-v^2)} \approx$$

$$\approx (2L_1+L_2+R)v \qquad (16)\ .$$

This identity shows that the described test would be called to measure at least a *first order effect* on v (v/c). For 1 metre radius, a distance $L_1 = L_2$ equal to 10 metres, an angular speed ω corresponding to 100 Herz, according to SR one should have a "shifting effect", after just *one* reflection, of about 63 microns, which is perhaps a shifting not so small to start with, and then to be finally detected after many reflections.

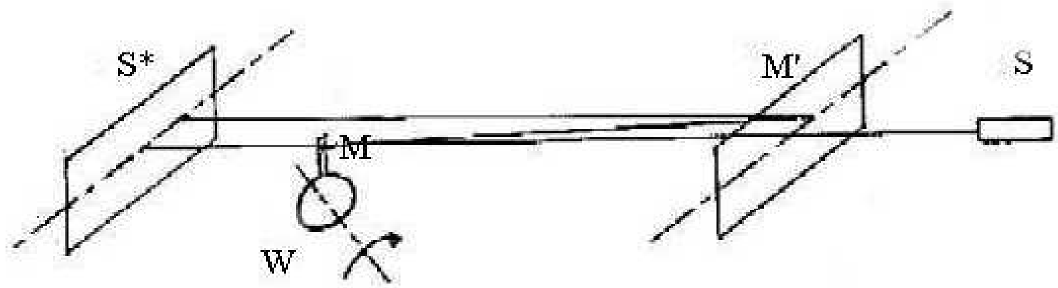

Figure 3: The proposed experiment, using
a spinning rotor W, as seen from a side.
S and S* as above
M and M' as above

It would be perhaps even better, from practical purposes, to replace the rotating platform with a "very sly" *vertical* spinning rotor - see figure 3 - in order to get always an angle of 90° between the forward beam, and the reflecting mirror M. In such a way, the proposed "experiment" would rather be similar to the famous *Fizeau cog-wheel* experiment.

Summing up, the light emitted by S is periodically reflected by M. Then, after a new reflection by M', the beam hits the mirror-detector S*, from which it is once again reflected to M', and so on. If relativity is correct, one should be able to appreciate an increasing *displacement* of the trace of the reflected photons, compared with the original trace, which corresponds to photons which have not been reflected. If an aether theory is correct, one should not observe any displacement. The *qualitative* side of the proposed experiment could perhaps be one of its most attractive features[29].

Endnotes

[1] And more than that, in strictly special-relativistic terms, since gravitation has obviously nothing to do with this question. For a discussion of the impossibility of a purely kinematical solution see for instance G. Cavalleri(6).

[2] We shall make use of *geometrical unities*, namely, we shall assume that the fundamental relativistic constant c (light's speed *in vacuo*) is equal to 1.

[3] Sometimes, one finds the paradox expressed in the form $L' = 2\pi R$, and then the "absurd" consequence would be now that a non-euclidean geometry holds in Ω!

[4] But one could say that there exists an *external synchronization* in P, induced by the privileged frame Ω (see also later on).

[5] For a quite rigorous introduction to this whole argument, see for instance B. O'Neill(9), pp. 358-360.

[6] $g = g_{33}$, in standard notations, is *a priori* any (negative) function of the coordinates (X,Y,T), and since $ds^2 = 0$ defines a photon's path, then $\sqrt{(-g)}$ is the *light's speed* in the given coordinate system (as a consequence, this speed can assume any value, even *bigger* than c = 1; see B. O'Neill(9), pp. 181-183). If the coordinate system is not orthogonal, then *space* and *time* are not separated enough, even from a relative point of view, in order to introduce the fundamental concepts of Physics.

[7] This is of course true under the relativistic assumption that only proper time is the time measured by clocks.

[8] Even just a "local" one, namely defined only on an *open* subset of **M**.

[9] In order to have *smooth* functions, one has to exclude in any case the value $\rho = 0$.

[10] From the point of view of SR, the "set of curves" (4) **is** the rotating platform, and this interpretation implies that all "physical" considerations about "materials" could be considered off the point.

[11] This cylinder **is** the rotating circumference in the Minkowski space-time **M,** in the same sense of endnote 10, since all curves (4), for $\rho = R$, "generate" this surface. As a consequence, one could say that the "rotation" **does not change** the circumference (or the platform) as *absolute objects* in space-time, the only difference is in the set of worldlines (observers) that one can introduce in order to generate the *same* surface.

[12] B. O'Neill[(9)], Prop. 30, p. 358. We remember that, if x^i are the Lorentz coordinates we are dealing with, and U_i the *covariant* components of the field U (with the usual symbolism), then U is said to be irrotational if the 2-form curl(U) = $d(U_i dx^i) = dU_i \wedge dx^i$ vanishes on each pair of vector fields V, W *orthogonal* to U: curl(U)(V,W) = 0. Vector fields such that curl(U) globally vanishes, are just the *geodesic* and irrotational ones, and this vanishing is obviously impossible in our case (with the further consequence that one could never introduce for our purposes, as we have already said, a *proper time synchronizable* coordinate system - or even a simple *synchronizable* one, whose associated observer field then should be at least irrotational).

[13] If one tries to make other circumferences of radius ρ rotate with the same speed v as the circumference of radius R, then he would not get an irrotational field. If one extends the field on C' by simple "parallel translation" in **M**, then he would get at last an irrotational field, but this cannot obviously include the "whole" of C'.

[14] As a matter of fact, one could notice that the "geometrical distance" between the event $(R\cos(\omega t^*), R\sin(\omega t^*), t^*)$ and the intersection of the rest-space (5) with the world-line (0,0,t) (this event is indeed $(0,0,t^*)$), is exactly equal to R, for each instant t*.

[15] This is just a "classical concept", and the root of all relativistic paradoxes is indeed the attempt to force such concepts into relativistic schemes!

[16] Namely, wrt to just *one* coordinate system.

[17] Which is defined by means of: $d\tau = \sqrt{(-ds^2)} = (\sqrt{(1-v^2)})dt$.

[18] Even if this should be obvious, let us remark that in SR only inertial (*geodesic*) observers cannot detect any possible "absolute motion", and that in our case A' is indeed non-inertial.

[19] Of course, we do not assert that this possible definition overcomes the previous objections: indeed, it has not been given by means of a coordinate system associated to C', or even simply to A'.

[20] In this case of the rotating circular platform, there is truly a great difference with the well-known relativistic uniform motion case (which one can easily study by means of Lorentz coordinates and transformations): the relativistic *average* light's velocities forward and backward could be defined in such a way to coincide with the "classically expected" values. A result which could appear "strange", and even more strange if one points out that the *instantaneous* light's speeds, as locally measured by *any* observer on C (by any coordinate system which can be "associated" to the given observer), **by means of its proper time**, always must give the same result, namely the *constant* c = 1. As a matter of fact, one should not forget indeed that the only one invariant in SR, namely in Minkowski space-time, is the ds^2, which allows for instance to simply define (measure) *proper times*. The transfer from proper times to *speeds*, understood as the *ratio*: space/time, requires the definition of *lengths*, and for that one needs, as we have already said, the introduction of particular orthogonal coordinate systems. In the case of a non-inertial coordinate system, the coordinate time T will not coincide in general with the proper time of the observers associated to the coordinate system. This implies that *average* measures of the light's speed in the given coordinate system, made wrt a *single* clock, could possibly not coincide with c (see also endnote 6). Moreover, let us remark again that in our case we have not introduced at all any coordinate system associated either to P' or to C', or to the observer A'!

[21] This could be considered a particular case of the famous *twin paradox* (for an up-to-date discussion of this old question, and of the so-called *Dingle's syllogism*, see U. Bartocci[(12)]), but one should indeed point out that only the non-inertial observer A' undergoes time dilation, wrt B, and not vice versa!

[22] But along this line of thought, one should be careful now in defining the radius of C', since the centre of P' would not be at rest for instance in $\Omega_0(t^*)$.

[23] With the present use of the terms "apparent" and "real", we wish to emphasize that all "proper quantities" in our context should be, contrarily to the common custom, the ones measured by the only one *privileged* frame which appears in this discussion, namely Ω, and not by some rotating observer on P'!

[24] Here we mean of course in the 2-dimensional geometrical sense. As a matter of fact, if we introduce the "tangent" IFR that we have called $\Omega_0(T)$, this direction coincides with the direction of the y^0-axis in $\Omega_0(T)$.

[25] For instance, in Italian we have just one word: "velocità", like in German one has only: "geschwindigkeit".

[26] And when v is "very small", one can directly approximate $\theta \approx v$, as usual.

[27] In U. Bartocci, M. Mamone Capria[(13)] it is suggested instead an experiment aimed to test the validity of the I SR Postulate (namely, of the Principle of Relativity) for electromagnetic interactions.

[28] The experimental set-up which has been just described is quite similar to the one used by P.A. Davies and R.C. Jennison[(14)], aimed to find possible evidence for a *Doppler transverse effect*.

[29] Needless to say, one should discuss many more practical *details* of this "experiment", such as the necessity to provide for thermic stability of the apparatus, etc., and more than all the problem of beam's angular resolution due to *diffraction*. From a theoretical point of view, instead, one should add some *hypothesis* about the possible "state" of a terrestrial laboratory wrt the "aether". The author believes that such a laboratory is very likely "in absolute rest" (*Stokes theory*), and so that it could be indeed a "privileged" (local) reference frame, but this discussion would obviously go quite far the limited present purpose of this paper...